SURFACES WITH CANONICAL MAP OF ODD DEGREE

MARGARIDA MENDES LOPES, RITA PARDINI, AND ROBERTO PIGNATELLI

Abstract. Let S be a smooth complex minimal surface of general type with $p_g := h^0(K_S) \geq 4$ whose canonical map is generically finite of odd degree $d > 1$ onto a surface Σ.

We assume that the general canonical curve of S is smooth and that Σ is ruled by lines, and we prove:

- $p_g \leq d + 2$
- Σ is a cone over the rational normal curve of degree $p_g - 2$ in $\mathbb{P}^{p_g-1}$
- $p_g = d + 2$ can occur only for $d = 3, 9, 11$.

As a byproduct, we refine previous results by Beauville and Xiao by proving that if one drops the assumption that Σ is ruled by lines then $d \leq 5$ if $p_g \geq 112$.

The case $d = 3$ being completely classified in [MP98], we focus on $d = 5$, showing that $p_g \leq 5$ and that for $p_g = 5$ the surface S has a pencil $|C|$ with $C^2 = 1$ and $K_S C = 5$.

These results suggest that the answer to the question whether the surfaces with canonical map of odd degree $d > 1$ have bounded invariants could be positive, in sharp contrast with the case of even degree.

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Contents

1. Introduction

Let S be a smooth minimal surface of general type with $p_g(S) := h^0(K_S) \geq 4$, let $\varphi\colon S \to \Sigma \subset \mathbb{P}^{p_g(S)-1}$ be the canonical map, and assume that the canonical image Σ is a surface. By results of Beauville [Be79] and Xiao

[Xi86], for $p_g(S) \gg 0$ the degree of φ is at most 8 and there are series of examples with canonical map of degree 8 and unbounded invariants. However, as noted in [MP23, § 5], only a few sporadic examples where φ is not birational of odd degree are known, and it is an open question whether such surfaces have bounded invariants. For degree 3 and Σ ruled by lines there is a complete classification under the additional assumption that the general canonical curve of S be smooth ([MP98], cf. also [St98]): $p_g(S) \leq 5$, Σ is a cone over the rational normal curve of degree $p_g(S) - 2$, the ruling of Σ pulls back to a linear pencil $|C|$ on S, there are only few possibilities for the triple $(p_g(S), C^2, K_S \cdot C)$, and all these possibilities actually occur.

Here we study the same situation for all odd values $d > 1$ of the degree of the canonical map, still under the assumption that the general canonical curve be smooth. The geometric picture that we obtain is very similar to the one for $d = 3$:

- $p_g(S) \leq d + 2$,
- Σ is a cone over the rational normal curve of degree $p_g(S) - 2$
- the ruling of Σ pulls back to a linear pencil $|C|$ on S and there is a short list of possibilities for the triple $(p_g(S), C^2, K_S \cdot C)$.

There is nevertheless a striking difference: not only we have not been able to find examples with $d > 3$, but we have ruled out the existence of most of the possible cases. In particular, for $d = 5$ we have worked out explicitly the list of possibilities for the triple $(p_g(S), C^2, K_S \cdot C)$, which is very similar to the one for $d = 3$. There are two "main" cases, $p_g = 7$, $C^2 = 1$, $K_S \cdot C = 5$ and $p_g = 5$, $C^2 = 2$, $K_S \cdot C = 6$, and one expects that the remaining cases can be obtained as specializations, e.g. by imposing elliptic singularities. However, in contrast with the case $d = 3$, we rule out the existence of both main cases and of most of the would-be specializations, so that the only remaining possibilities are $p_g = 5$, $C^2 = 1$, $K_S \cdot C = 5$ and $p_g = 4$.

This analysis is the core of the paper and we believe it to be of interest also from the point of view of the techniques: the proofs combine arguments on the numerical connectedness of curves on (possibly singular) surfaces, classification results for surfaces of small degree in projective spaces, the structure of the gap set of Weierstraß points on curves of small genus and the fact that the parity of theta characteristics is preserved in families. Actually, to exclude the case $p_g = 5$, $C^2 = 2$, $K_S \cdot C$ (the hardest!) we make use of an infinitesimal version of the preservation of the parity of theta characteristics [MPP26]. We refer the reader to §3 for a complete description of the results and an outline of the strategy of proof; we just mention here Corollary 3.3, that implies that surfaces with canonical map of odd degree ≥ 7 and smooth general canonical curve have bounded degree, improving results by Beauville [Be79] and Xiao [Xi86]. Summing up, the results contained in this paper seem to point to a positive answer to the question whether surfaces with non birational canonical map of odd degree have bounded invariants.

Finally, a comment on the assumption, made throughout all the paper, that the general canonical curve be smooth. On one hand it can be regarded as a mild restriction, since it is "almost always" satisfied in concrete examples, but on the other hand it would be very desirable to remove it and have unconditional statements. Unfortunately, this assumption is essential at all crucial points of our proofs, hence at the moment we have no clue on how to dispense with it.

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The first named author is a member of Centro de Análise Matemática, Geometria e Sistemas Dinâmicos. The second and third named authors are members of GNSAGA of INDAM.

1.2. **Notation and conventions.** We work over $\mathbb{C}$. Varieties embedded in a projective space are assumed to be non degenerate, unless otherwise stated. A *surface* X is a normal complex projective surface; when X is smooth or has canonical singularities we denote as usual by K_X the canonical class and write $p_g(X) := h^0(K_X)$ (*geometric genus*) and $q(X) := h^1(\mathcal{O}_X)$ (*irregularity*).

A *curve* D on a smooth surface S is a non-zero effective divisor (in particular D is Gorenstein).

Given a curve D, we denote its dualizing sheaf by ω_D and its arithmetic genus $p_a(D)$, where by definition $1 - p_a(D) = h^0(D, \mathcal{O}_D) - h^1(D, \mathcal{O}_D)$. A curve D is *m-connected* if for every decomposition $D = A + B$ as the sum of two curves A and B one has $AB \geq m$. A (-2)-*curve* (or *nodal curve*) on a smooth surface S is an irreducible curve that is isomorphic to $\mathbb{P}^1$ and has self-intersection (-2).

A pencil on a smooth surface S is a rational map with connected fibers $C \to B$, where B is a smooth curve. The pencil is said to be *linear* if $B = \mathbb{P}^1$, and *irrational of genus g* otherwise, where g is the genus of B.

2. Preliminaries

2.1. **Surfaces ruled by lines.** Let $\Sigma \subset \mathbb{P}^n$ be a non-degenerate surface and let $\rho\colon Y \to \Sigma$ be the minimal resolution of singularities. We say that Σ is *linearly normal* if the linear system $\rho^*|\mathcal{O}_\Sigma(1)|$ is complete.

We recall the following result from [CDM25]:

Theorem 2.1. *Let $\Sigma \subset \mathbb{P}^n$ be a surface ruled by lines. If Σ is linearly normal then:*

(i) *if* $q(\Sigma) = 0$*, then* $\deg \Sigma = n-1$ *and* Σ *is either a cone or a rational normal scroll;*
(ii) *if* $q(\Sigma) > 0$*, then* Σ *is a scroll with* $n - 1 + q(\Sigma) \leq \deg \Sigma \leq n - 1 + 2q(\Sigma)$ *and if* $\deg \Sigma = n - 1 + q(\Sigma)$ *then* Σ *is a cone.*

We will also use the following result obtained independently by Xiao Gang and Miles Reid:

Theorem 2.2. [Xi86, Lemma 1], [Re86] *Let* $Y \subset \mathbb{P}^n$ *be a non-degenerate surface such that* $\deg Y < \frac{4}{3}(n-2)$*.*

Then either Y *is ruled by lines or* $n = 9$ *and* $Y \subset \mathbb{P}^9$ *is the image of* $\mathbb{P}^2$ *via the complete system* $|\mathcal{O}_{\mathbb{P}^2}(3)|$*.*

2.2. **Weierstraß points on Gorenstein curves.** Let C be a Gorenstein curve with $h^0(\mathcal{O}_C) = 1$, and let p be a smooth point of C. Let g denote the arithmetic genus of C.

As for smooth curves, the Riemann-Roch Theorem and Serre Duality imply that $h^0(C, np) = n - g + 1$ for $n \geq 2g - 1$. Moreover $h^0(C, np) - h^0(C, (n-1)p) = 0$ or 1, so we can borrow the usual terminology and say that n is a gap for p if $h^0(C, np) = h^0(C, (n-1)p)$, and that p is a Weierstraß point if its set of gaps is not $\{1, \ldots, g-1\}$. Observe that as in the smooth case the cardinality of the set of gaps is g.

Lemma 2.3. *Let* C *be a Gorenstein curve with* $h^0(\mathcal{O}_C) = 1$*, and let* p *be a smooth point of* C*. Then the complement of the set of gaps of* p *is a semigroup.*

Proof. Let C_p be the irreducible component of C that contains p. Observe that the restriction map $H^0(C, np) \to H^0(C_p, np)$ is injective: a nonzero section in the kernel would be a nonzero section of $\mathcal{O}_C$ that vanishes on C_p, contradicting the assumption that $h^0(\mathcal{O}_C) = 1$. So if $s_i \in H^0(C, n_i p)$, $i = 1, 2$, are nonzero sections then their restrictions on C_p is nonzero; as C_p is irreducible then $s_1 s_2$ is also nonzero on C_p and then on C too. The statement follows from the same argument as in the case of smooth curves. □

If ω_C is a multiple of p we have the following easy lemma.

Lemma 2.4. *Let* C *be a Gorenstein curve of genus* $g \geq 2$ *with* $h^0(\mathcal{O}_C) = 1$*, and let* p *be a smooth point of* C*. Assume* $\omega_C \cong \mathcal{O}_C((2g-2)p)$*.*

Then p *is a Weierstraß point of* C *with minimal gap* 1 *and maximal gap* $2g-1$*. The remaining* $g-2$ *gaps are one for each pair* $\{2, 2g-3\}$*,* $\{3, 2g-4\}$*,* $\{4, 2g-5\}$*, …,* $\{g-1, g\}$*.*

Proof. Since $\omega_C = \mathcal{O}_C((2g-2)p)$ then $2g-1$ is a gap, the maximal possible gap. The Riemann-Roch Theorem and Serre Duality imply that there is exactly one gap on each of the $g-1$ pairs $\{1, 2g-2\}$, $\{2, 2g-3\}, \ldots, \{g-1, g\}$. In fact 1 is obviously a gap, i.e. the gap of the pair $\{1, 2g-2\}$. □

2.3. **Parity of theta characteristics.** We recall a generalized version due to Harris ([Ha82, Theorem 1.10.(i)], see also [Co89]) of the classical fact (cf. [Mu71]) that the parity of theta characteristic is constant in families.

Theorem 2.5. [Ha82, Theorem 1.10.(i)] *Let Δ be an irreducible variety, let $\pi\colon X \to \Delta$ be a proper flat map with fibers $C_t := \pi^{-1}(t)$ reduced curves, let $\mathcal{L}$ be a line bundle on X and set $\mathcal{L}_t := \mathcal{L}|_{C_t}$.*

If $\mathcal{L}_t^{\otimes 2} \cong \omega_{C_t}$ for all $t \in \Delta$, then the function $t \mapsto h^0(C_t, \mathcal{L}_t)$ is constant modulo 2.

We also need the infinitesimal version of Theorem 2.5 from [MPP26, Corollary 1.2]

Theorem 2.6. *In the assumptions of Theorem 2.5, suppose in addition that Δ is a smooth curve.*

Then

(i) *for all $t \in \Delta$, for all $k \in \mathbb{N}$, $h^0(kC_t, \mathcal{L}|_{kC_t})$ equals $kh^0(C_t, \mathcal{L}|_{C_t})$ modulo* 2*;*
(ii) *there is a coherent sheaf $\mathcal{T}$ on Δ such that the torsion subsheaf of $R^1\pi_*\mathcal{L}$ is isomorphic to $\mathcal{T} \oplus \mathcal{T}$.*

3. Statement of the results

The next theorem summarizes our main results.

Theorem 3.1. *Let S be a minimal surface of general type with $p_g(S) \geq 4$ such that the canonical map $\varphi\colon S \to \mathbb{P}^{p_g(S)-1}$ is generically finite. Assume that:*

(a) the general canonical curve $D \in |K_S|$ is smooth;
(b) the degree d of φ is odd;
(c) the canonical image Σ is ruled by lines.

Then:

(i) *$p_g(S) \leq d + 2$ and Σ is the cone over the rational normal curve of degree $p_g(S) - 2$;*
(ii) *the strict transforms of the ruling of Σ induce a linear pencil of non hyperelliptic curves $|C|$ with $C^2 > 0$ and $g(C) \geq \frac{d+3}{2}$;*
(iii) *φ separates the curves of $|C|$;*
(iv) *if $p_g(S) = d + 2$, then $d = 3, 9, 11$, the system $|K_S|$ is base point free, $C^2 = 1$ and $K_S = dC$;*
(v) *if $d = 5$, then $p_g(S) \leq 5$; if $p_g(S) = 5$ then $C^2 = 1$, $K_SC = 5$.*

The proof takes up the rest of the paper: proving (iv) and (v) is by far the hardest part.

In the next section we prove Proposition 4.1, which is a weaker version of Theorem 3.1. More precisely Proposition 4.1 proves (i), (ii), (iii) and a weaker version of (iv), namely that if $p_g(S) = d+2$, then $d \leq 11$, the system $|K_S|$ is base point free, $C^2 = 1$ and $K_S = dC$. The proof of (iv) is completed by excluding that $p_g(S) = d+2$ occur for $d = 5$ or 7: this is Proposition 7.1. As a consequence $d = 5$ implies $p_g \leq 6$: (v) then follows by Propositions 5.3, 7.7 and 7.18.

The conclusion of Theorem 3.1 holds also if one assumes that Σ is a surface of minimal degree $p_g - 2$ in $\mathbb{P}^{p_g-1}$. In fact by Theorem 2.2 the only surface of minimal degree which is not ruled by lines is the Veronese surface $V_{2,2} \subset \mathbb{P}^5$ and we have the following:

Proposition 3.2. *Let S be a minimal surface of general type such that the general canonical curve $D \in |K_S|$ is smooth. If the canonical image of S is the Veronese surface $V_{2,2} \subset \mathbb{P}^5$ then the degree d of the canonical map φ of S is even.*

Proof. Let C be the pull back via φ of a line in $\mathbb{P}^2 \cong V_{2,2}$, so that $K_S = 2C$. If φ is not a morphism, then $|C|$ has at least a base point $x \in X$. Since $|K_S| = \varphi^*|\mathcal{O}_{\mathbb{P}^2}(2)|$ and the map $\operatorname{Sym}^2 H^0(\mathcal{O}_{\mathbb{P}^2}(1)) \to H^0(\mathcal{O}_{\mathbb{P}^2}(2))$ is surjective, it follows that x is a base point of $|K_S|$ of multiplicity at least 2, namely the general canonical curve is singular at x, contradicting the assumptions. So φ is a morphism and $C^2 = d$. We conclude by observing that $C^2 = C(K_S - C)$ is even by the adjunction formula. $\square$

Theorem 3.1 has the following almost immediate consequence:

Corollary 3.3. *Let S be a minimal surface of general type such that the canonical map $\varphi\colon S \to \mathbb{P}^{p_g(S)-1}$ is generically finite. Assume that:*

(a) the general canonical curve $D \in |K_S|$ is smooth;
(b) the degree d of φ is odd;

Then:

- *if $d = 7$ then $p_g \leq 111$;*
- *if $d = 9$ then $p_g \leq 15$;*
- *if $d = 11$ then $p_g \leq 13$;*
- *if $d \geq 13$ then $p_g \leq 2 + \frac{27}{d-9}$.*

Proof. As in Theorem 3.1, we denote by Σ the canonical image $\varphi(S)$. By the Bogomolov-Miyaoka-Yau inequality

$$d \deg \Sigma \leq K_S^2 \leq 9(p_g + 1) \tag{3.1}$$

and $\deg \Sigma \geq p_g - 2$ we obtain that

$$p_g \leq \frac{2d+9}{d-9} = 2 + \frac{27}{d-9} \text{ when } d \geq 10. \tag{3.2}$$

If $d = 11$, (3.1) implies $\deg \Sigma \leq \frac{9}{11}(p_g + 1)$, which is, for $p_g \geq 10$, smaller than $\frac{4}{3}(p_g - 3)$: then by Theorem 2.2, the surface Σ is ruled by lines and Theorem 3.1 gives $p_g \leq d + 2 = 13$.

If $d = 9$, (3.1) implies $\deg \Sigma \leq p_g + 1$, which is, for $p_g \geq 16$, smaller than $\frac{4}{3}(p_g - 3)$, forcing Σ to be ruled the lines: then Theorem 3.1 gives $p_g \leq 11$, a contradiction. The same argument proves that $d = 7$ implies $p_g \leq 111$. $\square$

Remark 3.4. *Xiao Gang (*[Xi86]*) proved that if the canonical map of a surface has degree* 9 *then* $p_g \leq 132$*; assuming that the general canonical curve be smooth we obtain a much stronger bound. Moreover our assumption allows us to get a bound also for* $d = 7$*.*

4. General properties

Here we prove a partial version of Theorem 3.1.

More precisely, note first that Proposition 4.1 below has exactly the same assumptions as Theorem 3.1. Moreover the claims (i),(ii),(iii) of Proposition 4.1 are the same as in Theorem 3.1, whereas claim (iv) of Proposition 4.1 is weaker than claim (iv) of Theorem 3.1.

Proposition 4.1. *Let* S *be a minimal surface of general type with* $p_g(S) \geq 4$ *such that the canonical map* $\varphi\colon S \to \mathbb{P}^{p_g(S)-1}$ *is generically finite. Assume that:*

- *(a) the general canonical curve* $D \in |K_S|$ *is smooth;*
- *(b) the degree* d *of* φ *is odd;*
- *(c) the canonical image* Σ *is ruled by lines.*

Then:

- (i) $p_g(S) \leq d+2$ *and* Σ *is the cone over the rational normal curve of degree* $p_g(S)-2$*;*
- (ii) *the strict transforms of the rulings of* Σ *induce a linear pencil of non hyperelliptic curves* $|C|$ *with* $C^2 > 0$ *and* $g(C) \geq \frac{d+3}{2}$*;*
- (iii) φ *separates the curves of* $|C|$*;*
- (iv) *if* $p_g(S) = d+2$*, then* $d \leq 11$*, the system* $|K_S|$ *is base point free,* $C^2 = 1$ *and* $K_S = dC$*.*

Proof. Set $p_g(S) =: n+1$. Let $\varepsilon\colon \widehat{S} \to S$ be the blow up at the, possibly infinitely near, base points of $|K_S|$, and denote by $E_1, \dots E_k$ the corresponding, possibly reducible, (-1)-curves. We denote by $\widehat{\varphi}\colon \widehat{S} \to \mathbb{P}^n$ the morphism induced by φ. Since the general canonical curve is smooth by assumption, the moving part of $|K_{\widehat{S}}|$ is $M := K_{\widehat{S}} - 2\sum E_i$.

Let $h\colon \widehat{S} \to B$ be the pencil (with connected fibers) induced by the ruling of the canonical image and denote by $\widehat{C}$ a general fiber of h. The strict transform G of a general line of Σ is numerically equivalent to $\delta\widehat{C}$, where δ is a divisor of d, so $G^2 = \delta^2\widehat{C}^2 \geq 0$. In addition $MG = d$, so $K_{\widehat{S}}G = d + 2\sum E_i G$ is odd and the adjunction formula gives $G^2 > 0$, hence also $\widehat{C}^2 > 0$. So h is not a morphism, $B = \mathbb{P}^1$ and Σ is a cone over a rational curve.

By Theorem 2.1, Σ has degree $n-1$ and it is the cone over the rational normal curve of degree $n-1$. The Hodge index theorem gives:

$$G^2(d(p_g(S)-2)) = G^2M^2 \leq (MG)^2 = d^2$$

from which

$$p_g(S) \leq 2 + \frac{d}{G^2} \leq d + 2. \tag{4.1}$$

We have then proved (i),

Assume that $\widehat{\varphi}$ does not separate the curves of $|\widehat{C}|$, namely that $\delta > 1$. The assumption $p_g \geq 4$ and (3.1) imply $d \leq 22$. Since $G^2 \geq \delta^2$, then (4.1) implies $\delta < 5$. Since δ is odd, we obtain $\delta = 3$. Then (4.1) gives $d \geq 18$. On the other hand d is odd, divisible by $\delta = 3$ and not greater than 22, so $d = 21$. This implies, using (4.1) again, $p_g = 4$. Then pulling back a hyperplane section through the vertex of the quadric cone Σ we find that $6\widehat{C} \leq K_{\widehat{S}}$, which implies $p_g \geq 7$, a contradiction. This shows that φ separates the curves of $|C|$, proving (iii).

Then $G = \widehat{C}$ and $2g(\widehat{C}) - 2 = \widehat{C}^2 + K_{\widehat{S}}\widehat{C} \geq 1 + K_{\widehat{S}}\widehat{C} \geq 1 + M\widehat{C} = 1 + d$, so $g(C) \geq g(\widehat{C}) \geq \frac{d+3}{2}$, completing the proof of (ii).

Now we assume $p_g(S) = d + 2$. The Bogomolov-Miyaoka-Yau inequality (3.1) gives $d \leq 11$. The Index Theorem gives:

$$d^2 = d\,(p_g(S) - 2) = d\,(\deg \Sigma) = M^2 \leq \widehat{C}^2 M^2 \leq (\widehat{C}M)^2 = d^2,$$

so $\widehat{C}^2 = 1$ and $M \sim_{num} d\widehat{C}$. On the other hand Σ is the cone over the rational normal curve of degree d, so $M = d\widehat{C} + Z$, with $Z \geq 0$. Comparing the two expressions for M we get $Z \sim_{num} 0$ and therefore $Z = 0$ and $M = d\widehat{C}$. Now assume by contradiction that $|K_S|$ has base points, namely $\widehat{S}$ contains a (-1)-curve E; since we assumed the general canonical curve to be smooth, all the base points of $|K_S|$ are simple, hence $ME = 1$, contradicting the fact that M is divisible by d in $\mathrm{Pic}(S)$. So $\widehat{S} = S$ and $dC = M = K_S$. □

5. Some more results on the case $d = 5$

To study the cases of Theorem 3.1 with $p_g \leq d+1$ we analyze some special canonical divisors on S, borrowing some arguments from [MP98].

5.1. **A useful canonical divisor.** In the situation of Proposition 4.1, we consider a general point z in the cone $\Sigma \subset \mathbb{P}^{p_g(S)-1}$ and enumerate as $x_1, \ldots, x_d$ the points of $\varphi^{-1}(z)$. If H is a general hyperplane through z, then $H_{|\Sigma}$ is a smooth rational curve whose pull-back on S is a smooth canonical curve D. Then $h^0(D, x_1 + \ldots + x_d) \geq h^0(\mathcal{O}_{\mathbb{P}^1}(1)) = 2$. By Serre duality, $x_1, \ldots, x_d$ do not impose independent conditions on the canonical system of D and therefore, a fortiori, on the linear system $|2K_S|$ on S.

Consider the pencil $|C|$ on S induced by the ruling of Σ. Since Σ is a cone, for any general point z of Σ there is a canonical curve D_z on S such that $D_z = 2C_z + B$, where C_z is the fibre of $|C|$ corresponding to the line through z, $B \geq 0$ and all points $x_1, \ldots, x_d$ belong to C_z and not to B.

In this situation we can use [MP98, Lemma 2.3], that we recall here.

Lemma 5.1. *Let S be a smooth surface, let D be a curve on S and let $x_1, \ldots, x_d$ be distinct singular points of D. Let $p\colon \hat{S} \to S$ be the blow-up at $x_1, \ldots, x_d$ and let $E_1, \ldots, E_d$ be the corresponding exceptional curves. Setting $D' = p^*D - E_1 - \ldots - E_d$ and $D'' = p^*D - 2E_1 - \ldots - 2E_d$, the following two conditions are equivalent:*

(i) *the points $x_1, \ldots, x_d$ do not impose independent conditions on the linear system $|K_S + D|$;*
(ii) *the restriction map $H^0(D', \mathcal{O}_{D'}) \to H^0(D'', \mathcal{O}_{D''})$ is not surjective.*

We apply the above to D_z. Since D_z is 1-connected, it is not hard to check that D' is also 1-connected and so $h^0(D', \mathcal{O}_{D'}) = 1$. Since $x_1, ..., x_d$ do not impose independent conditions to $|2K_S|$, by Lemma 5.1 $h^0(D'', \mathcal{O}_{D''}) \geq 2$. Note that, by our choice of D_z, none of the E_i is a component of D''. Applying Lemma A.1 of [CFM97] to the sheaf $\mathcal{O}_{D''}$, we obtain a decomposition $D'' = A_1 + A_2$ with A_1, A_2 effective non-zero divisors such that $A_1A_2 \leq 0$ and every component θ of A_1 satisfies $\theta A_2 \leq 0$.

Reasoning as in the proof of Theorem 5 of [Bo73], this decomposition of D'' yields in turn a decomposition of D_z as $D_z = D_1 + D_2$ where $D_1D_2 = m \leq d$ and at least m of the points $x_1, ..., x_d$ belong to both D_1 and D_2. Note that $m \geq 2$, by the 2-connectedness of the canonical curves. Furthermore, any component θ of D_1 not passing through the points $x_1, ..., x_d$ satisfies $\theta D_2 \leq 0$. By our choice of D_z we conclude that C_z is a common component of D_1 and D_2 and that every other component θ (if any) of D_1 satisfies $\theta D_2 \leq 0$. Furthermore, since $K_SC \geq d$, $K_SD_i \geq d$ for $i = 1, 2$.

Since D_z is a canonical curve, m is even. If d is odd, $m < d$ and so $D_i^2 + m = K_SD_i \geq d$ implies that $D_i^2 \geq d - m > 0$, for $i = 1, 2$.

In conclusion, we have

Proposition 5.2. *In the assumptions of Theorem 3.1, let z be a general point of Σ and let $x_1, \ldots, x_d$ be the preimages of z via the canonical map.*

There there is a canonical divisor D on S with a decomposition $D = D_1 + D_2$ such that the even number $m = D_1D_2$ is strictly smaller than d, and at least m of the points x_j are simple points of D_1; furthermore any component θ of D_1 not passing through the points $x_1, \ldots, x_d$ satisfies $\theta D_2 \leq 0$.

Let C_z be the element of the pencil $|C|$ on S induced by the ruling of Σ containing the x_j. Then

(i) *$C_z \leq D_1 \leq D_2$;*
(ii) *$1 \leq C^2 \leq D_1^2 \leq D_2^2$;*
(iii) *if $D_1 - C_z > 0$, then $D_1^2 \geq C^2 + 2$.*

Proof. From the explanations above we only have to prove (ii), (iii) and that $D_1 \leq D_2$.

Set G for the greatest common divisor of D_1 and D_2, and write $G_j = D_j - G$, for $j = 1, 2$. We know that all components θ of D_1 different from C_z do satisfy $\theta D_2 \leq 0$, and so $G_1D_2 \leq 0$. Since G_1 and G_2 have no common components, $G_1G_2 \geq 0$. So it follows $G_1(K_S - G_1) = G_1(2D_2 - G_2) \leq 0$. The 2−connectedness of every canonical divisor implies $G_1 = 0$. Hence $D_1 \leq D_2$.

Then, by the nefness of K_S, $D_2^2 = D_1^2 + 2D_1G_2 + G_2^2 = D_1^2 + K_SG_2 \geq D_1^2$.

Recall that $C^2 \geq 1$ by Proposition 4.1, (ii). If $D_1 = C_z$, (ii) is proved.

Else, $D_1 \neq C_z$. We write $D_1 = C_z + P$, with $P \neq 0$. Since the only component of D_1 through the points x_j is C_z then $PD_2 \leq 0$. Now $P(K_S - P) = PC_z + PD_2 \leq PC_z$. So by the 2-connectedness of canonical divisors we have $PC_z \geq 2$.

Since K_S is nef we have $PD_1 = PK_S - PD_2 \geq 0$. On the other hand $D_1^2 = C_zD_1 + PD_1 = C_z^2 + PC_z + PD_1$ and so $D_1^2 \geq C_z^2 + 2$. □

5.2. **Application to the case $d = 5$.** The previous results can be used to determine the numerical possibilities for C^2, K_SC and $p_g(S)$ in the situation of Theorem 3.1 for every value of d. Below we do it for $d = 5$, since this is the only case we analyze in greater detail in the paper.

Proposition 5.3. *In the assumptions of Theorem 3.1, let $|C|$ be the pencil on S induced by the ruling of Σ.*

Assume $d = 5$. Then

(i) *if $p_g \geq 6$ then $C^2 = 1$, $K_SC = 5$;*
(ii) *if $p_g = 5$ then either $C^2 = 1$, $K_SC = 5$ or $C^2 = 2$, $K_SC = 6$. In the latter case $K_S = 3C$.*

Proof. We consider the canonical divisor $D = D_1 + D_2$ from Proposition 5.2. Then D_1D_2 is either 2 or 4. In addition $C_z \leq D_1$, so $K_SD_1 \geq K_SC_z \geq 5$. If $D_1D_2 = 2$, then $5 \leq K_SD_1 = D_1^2 + D_1D_2$ gives $D_1^2 \geq 3$, contradicting $D_1^2 \leq D_2^2$ (Proposition 5.2) and the index theorem. So $D_1D_2 = 4$.

Then $5(p_g - 2) \leq K_S^2 = D_1^2 + D_2^2 + 8$, so $D_1^2 + D_2^2 \geq 5p_g - 18$. By the index theorem $D_1^2D_2^2 \leq 16$.

Assuming $p_g \geq 6$ this forces $D_1^2 = 1$ and then $D_1 = C_z$ by Proposition 5.2. It follows $C^2 = 1$, $K_SC = (D_1 + D_2)D_1 = 5$. We have proved (i).

If $p_g = 5$, then Σ has degree 3 and by construction $D = 2C_z + C_0 + Z$ where C_0 is an element of $|C|$ and $Z \geq 0$ is contracted by φ to the vertex of Σ. Since $D_1 \leq D_2$ then $D_2 \geq C_z + C_0$. Since K_S is nef

$$D_2^2 = K_SD_2 - D_1D_2 = K_SD_2 - 4 \geq 2K_SC - 4 \geq 10 - 4 = 6.$$

By the index theorem it follows $D_1^2 \leq \left\lfloor \frac{16}{6} \right\rfloor = 2$ and then $D_1 = C_z$ by Proposition 5.2. So either $C^2 = 1$, $K_SC = 5$ or $C^2 = 2$, $K_SC = 6$.

Finally, if $C^2 = 2$ then $K_SD_2 \geq 2K_SC = 12$ implies $D_2^2 \geq 8$, and then by the index theorem D_2 is numerically equivalent to $2C$, and then K_S is numerically equivalent to $3C$. So the effective divisor Z is numerically equivalent to 0 and so $Z = 0$, which concludes the proof. □

6. The linear systems $|mC|$

In this section we collect most of the technical results needed to prove points (iv) and (v) of Theorem 3.1.

Throughout all the section we will make the following:

Assumption–Notation 6.1. *S is a minimal surface of general type with an irreducible pencil $|C|$ such that $C^2 = 1$. In these assumptions the base locus of $|C|$ consists of a simple base point that we denote by p.*

Lemma 6.2. *Let $m > 0$ be an integer. Then:*

(i) *$h^0(mC) \geq m + 1$;*
(ii) *if $h^0(mC) \geq m+2$, then $h^0((m+h)C) \geq m+2h+2$ for all integers $h \geq 0$.*

Proof. (i) Fix $C \in |C|$ and an integer $k > 0$, and consider the exact sequence:

$$0 \to H^0((k-1)C) \to H^0(kC) \xrightarrow{r_k} H^0(\mathcal{O}_C(kp)). \tag{6.1}$$

Let $\sigma \in H^0(C)$ be a section that cuts out p on C. Then $r_k(\sigma^k) \neq 0$, hence the rank ρ_k of r_k is > 0 for all $k > 0$. Set $\bar{\sigma} := r_1(\sigma) \in H^0(\mathcal{O}_C(p))$; multiplication by $\bar{\sigma}$ gives an injection $\operatorname{Im} r_k \hookrightarrow \operatorname{Im} r_{k+1}$, hence ρ_k is a non decreasing function of $k \geq 1$. Since $h^0(mC) = 1 + \sum_1^m \rho_k$, we have $h^0(mC) \geq m+1$ with equality holding if and only if $\rho_k = 1$ for all $1 \leq k \leq m$. In particular this proves (i).

(ii) If $h^0(mC) \geq m+2$, then the above remarks imply $\rho_m \geq 2$ and $\rho_k \geq 2$ for all $k \geq m$, hence $h^0((m+h)C) = h^0(mC) + \sum_{j=1}^h \rho_{m+j} \geq m+2+2h$. □

Proposition 6.3. *In the above setup, let $m_0 > 0$ be an integer such that $h^0(\mathcal{O}_C(m_0p)) = 2$ and $h^0(\mathcal{O}_C((m_0-1)p)) = 1$ for all curves $C \in |C|$. Then*

$$h^0(m_0C) = m_0 + 2.$$

Proof. Let $\epsilon\colon \tilde{S} \to S$ be the blow up of S at p, and let E be the exceptional curve. Let $\tilde{f}\colon \tilde{S} \to \mathbb{P}^1$ be the fibration induced by $|C|$, so that C pulls back to $E+F$, F being a fibre of $\tilde{f}$. Note that each fibre F is isomorphic to the corresponding curve C and E cuts on F the point mapping to p.

Applying $\tilde{f}_*$ to the exact sequence

$$0 \to \mathcal{O}_{\tilde{S}}((m-1)E) \to \mathcal{O}_{\tilde{S}}(mE) \to \mathcal{O}_E(mE) \to 0$$

we obtain, for all $m \in \mathbb{N}$, the exact sequence

$$\begin{aligned} 0 \to \tilde{f}_*(\mathcal{O}_{\tilde{S}}((m-1)E)) \to \tilde{f}_*(\mathcal{O}_{\tilde{S}}(mE)) \to \mathcal{O}_{\mathbb{P}^1}(-m) \to \\ \to R^1\tilde{f}_*(\mathcal{O}_{\tilde{S}}((m-1)E)) \to R^1\tilde{f}_*(\mathcal{O}_{\tilde{S}}(mE)) \to 0 \end{aligned} \tag{6.2}$$

By Riemann-Roch the assumptions imply that $h^1(\mathcal{O}_F((m_0-1)p)$ is equal to $h^1(\mathcal{O}_F(m_0p))$ and is independent of the fiber F. Therefore the sheaves $R^1\tilde{f}_*(\mathcal{O}_{\tilde{S}}((m_0-1)E))$ and $R^1\tilde{f}_*(\mathcal{O}_{\tilde{S}}(m_0E))$ are locally free of the same rank. Recall that a surjective morphism of locally free sheaves of the same rank is an isomorphism. Then the map $R^1\tilde{f}_*(\mathcal{O}_{\tilde{S}}((m_0-1)E)) \to R^1\tilde{f}_*(\mathcal{O}_{\tilde{S}}(m_0E))$ is an isomorphism and we have a short exact sequence

$$0 \to \tilde{f}_*(\mathcal{O}_{\tilde{S}}((m_0-1)E)) \to \tilde{f}_*(\mathcal{O}_{\tilde{S}}(m_0E)) \to \mathcal{O}_{\mathbb{P}^1}(-m_0) \to 0$$

Note that $\tilde{f}_*(\mathcal{O}_{\tilde{S}}((m_0-1)E)) \cong \mathcal{O}_{\mathbb{P}^1}$, because it is a line bundle on $\mathbb{P}^1$ whose space of global sections has dimension 1, and then the above short exact sequence becomes

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to \tilde{f}_*(\mathcal{O}_{\tilde{S}}(m_0E)) \to \mathcal{O}_{\mathbb{P}^1}(-m_0) \to 0$$

which splits, because $h^1(\mathcal{O}_{\mathbb{P}^1}(m_0)) = 0$. So

$$\tilde{f}_*(\mathcal{O}_{\tilde{S}}(m_0E)) \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-m_0). \tag{6.3}$$

By (6.3) $h^0(\mathcal{O}_S(m_0C)) = h^0(\mathcal{O}_{\tilde{S}}(m_0(E+F))) = h^0(\tilde{f}_*\mathcal{O}_{\tilde{S}}(m_0(E+F))) = h^0(\mathcal{O}_{\mathbb{P}^1}(m_0) \oplus \mathcal{O}_{\mathbb{P}^1}) = m_0 + 2$. □

We note the following consequence of Proposition 6.3:

Corollary 6.4. *If $K_SC = 5$, $h^0(\mathcal{O}_C(K_S - 3C)) > 0$ and $h^0(\mathcal{O}_C(3p)) = 1$ for every curve $C \in |C|$, then $h^0(4C) = 6$.*

Proof. The curves of $|C|$ have genus 4 by the adjunction formula. If $C \in |C|$ is general then $K_C - 4p = (K_S - 3C)|_C$ is effective, hence $h^0(\mathcal{O}_C(4p)) \geq 2$ by Riemann–Roch. By semicontinuity $h^0(\mathcal{O}_C(4p)) \geq 2$ for every $C \in |C|$. On the other hand $h^0(\mathcal{O}_C(4p)) \leq h^0(\mathcal{O}_C(3p)) + 1 \leq 2$ for every $C \in |C|$, so $h^0(\mathcal{O}_C(4p)) = 2$ for every C and Proposition 6.3 gives $h^0(4C) = 6$. □

The next result is crucial in the proof of points (iv) and (v) of Theorem 3.1.

Proposition 6.5. *Assume that p is not a base point of $|K_S|$ and that for every $C \in |C|$ one has $K_S|_C = \mathcal{O}_C(dp)$, where $d = 5$ or $d = 7$. Then*

$$h^0\left(\frac{d+3}{2}C\right) \geq \frac{d+7}{2}.$$

The proof requires some intermediate results and takes up the rest of the section.

Lemma 6.6. *In the assumptions of Proposition 6.5 fix an element C of the pencil $|C|$. Then:*

(i) *if $d = 5$ the set of gaps of p in C is either $\{1, 2, 3, 7\}$ or $\{1, 2, 4, 7\}$;*
(ii) *if $d = 7$ the set of gaps of p in C is either $\{1, 2, 3, 4, 9\}$ or $\{1, 2, 3, 5, 9\}$.*

Proof. (i) By Lemma 2.4, if $d = 5$ then p has 4 gaps in C, the minimal one is 1 and the maximal one is 7 and there is exactly one gap in each of the pairs $\{2, 5\}$ and $\{3, 4\}$. Since $K_S|_C = \mathcal{O}_C(dp)$ and p is not a base point of $|K_S|$ by assumption, the point p is not a base point of $|\mathcal{O}_C(5p)|$ hence 2 is a gap. So the possible sets of gaps are $\{1, 2, 3, 7\}$ and $\{1, 2, 4, 7\}$.

(ii) By Lemma 2.4 if $d = 7$, p has 5 gaps in C, the minimal one is 1 and the maximal one is 9. Since the complement of the set of gaps is a semigroup and 3 divides 9, 3 is a gap too. Moreover the remaining two gaps are one in each pair $\{2, 7\}$, $\{4, 5\}$, and arguing as in case (i) one sees that 2 is a gap. So the set of gaps is either $\{1, 2, 3, 4, 9\}$ or $\{1, 2, 3, 5, 9\}$. $\square$

By the invariance of the parity of theta characteristics we observe

Lemma 6.7. *In the assumptions of Proposition 6.5 the set of gaps of p in C does not depend on the choice of the curve in the pencil.*

Proof. Consider again the blow up $\epsilon\colon \tilde{S} \to S$ of S at p. Let E be the exceptional divisor and let $\tilde{f}\colon \tilde{S} \to \mathbb{P}^1$ be the fibration induced by $|C|$, so that C pulls back to $E + F$, where F is a fibre of $\tilde{f}$. The map ϵ induces an isomorphism of each fibre F with the corresponding curve C and E cuts on F the point mapping to p.

For $d = 5$, in the two cases of Lemma 6.6, the theta characteristic $\mathcal{O}_C(3p)$ has different parity, since $h^0(\mathcal{O}_C(3p)) = 1$ or 2 respectively. Similarly, for $d = 7$ the theta characteristic $\mathcal{O}_C(4p)$ has different parity for the two possible sets of gaps.

We cannot conclude just applying Harris Theorem 2.5 to $\tilde{f}$, because the fibres of $\tilde{f}$ may a priori have irreducible components that are not reduced. We solve this issue by contracting some (-2)-curves of S so that we have a fibration with irreducible fibers, as follows. Every C has a distinguished component C_p, the one containing p (as a smooth point!), that is reduced and satisfies $K_S C_p = d$. All the remaining components A of C do not go through p, so they satisfy $K_S A = 0$ and are (-2)-curves that pull back to (-2)-curves of $\tilde{S}$ contained in the fibers of $\tilde{f}$. It is immediate to check that, since S is minimal, all the (-2)-curves contracted by $\tilde{f}$ arise in this way.

By contracting the (-2)-curves in the fibers of $\tilde{S}$, we obtain a surface $\bar{S}$ and a commutative diagram

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\ \pi\ } & \bar{S} \\ & {\scriptstyle \tilde{f}}\searrow \quad \swarrow{\scriptstyle \bar{f}} & \\ & \mathbb{P}^1 & \end{array}$$

The fibres $\bar{F} = \pi(F)$ of $\bar{f}$ are all irreducible and reduced, images of the distinguished components C_p. So we can apply Harris Theorem 2.5 to $\bar{f}$ obtaining that the parity of $h^0(\mathcal{O}_{\bar{F}}(4\pi(E)))$ does not depend on the fibre.

We conclude the proof by showing that $h^0(\mathcal{O}_{\bar{F}}(4\pi(E)))$ equals $h^0(\mathcal{O}_C(4p))$ for the corresponding curve C in the pencil.

Fix a fibre F and then its image $\bar{F} = \pi(F)$. Applying the functor π_* to the exact sequence

$$0 \to \mathcal{O}_{\tilde{S}}(-F) \to \mathcal{O}_{\tilde{S}} \to \mathcal{O}_F \to 0$$

we obtain the exact sequence

(6.4)

$$0 \to \pi_*\mathcal{O}_{\tilde{S}}(-F) \to \pi_*\mathcal{O}_{\tilde{S}} \to \pi_*\mathcal{O}_F \to R^1\pi_*\mathcal{O}_{\tilde{S}}(-F) = R^1\pi_*\mathcal{O}_{\tilde{S}} \otimes \mathcal{O}_{\mathbb{P}^1}(-1)$$

Note that $R^1\pi_*\mathcal{O}_{\tilde{S}}$ vanishes because the singularities of $\bar{S}$ are rational by construction and π is the minimal resolution. Since $F = \pi^*\bar{F}$, applying the projection formula we rewrite (6.4) as

$$0 \to \mathcal{O}_{\bar{S}}(-\bar{F}) \to \mathcal{O}_{\bar{S}} \to \pi_*\mathcal{O}_F \to 0$$

from which

$$\pi_*\mathcal{O}_F \cong \mathcal{O}_{\bar{F}}$$

Then by the projection formula $\pi_*\mathcal{O}_F(4E) = \pi_*\mathcal{O}_F(\pi^*(4\pi(E)) \cong \mathcal{O}_{\bar{F}}(4\pi(E))$.

The proof is completed by remarking that the map ϵ induces isomorphisms $\mathcal{O}_C(4p) \cong \mathcal{O}_F(4E)$. □

Proof of Proposition 6.5. By Lemma 6.7, the set of gaps of p in C does not depend on the choice of the curve in the pencil, so if m_0 is the smallest non gap then $h^0(m_0D) = m_0 + 2$ by Proposition 6.3. Since $m_0 \leq \frac{d+3}{2}$ by Riemann-Roch, Proposition 6.3 also gives the inequality $h^0(\frac{d+3}{2}C) \geq \frac{d+7}{2}$. □

7. Proof of Theorem 3.1, (iv) and (v)

By Proposition 4.1 and Proposition 5.3 the possible cases for $d = 7$ and $p_g = 9$ and for $d = 5$ and $p_g \geq 5$ are the following:

- $d = 7$, $p_g = 9$, $K_SC = 7$, $C^2 = 1$;
- $d = 5$, $p_g = 7$, $K_SC = 5$, $C^2 = 1$;
- $d = 5$, $p_g = 6$, $K_SC = 5$, $C^2 = 1$;
- $d = 5$, $p_g = 5$, $K_SC = 6$, $C^2 = 2$;
- $d = 5$, $p_g = 5$, $K_SC = 5$, $C^2 = 1$.

In order to complete the proof of Theorem 3.1 we have to rule out all possibilities except the last one. We start with the cases where p_g has the maximum value $d + 2$.

Proposition 7.1. *The cases $d = 5$, $p_g = 7$ and $d = 7$, $p_g = 9$ of Proposition 4.1 do not occur.*

Proof. By Proposition 4.1 if $p_g = d + 2$ then $C^2 = 1$, $K_S = dC$ and $|K_S|$ is base point free. So Proposition 6.5 applies, giving $h^0(4C) \geq 6$ if $d = 5$ and $h^0(5C) \geq 7$ if $d = 7$. By Lemma 6.2 if $d = 5$ we get $7 = p_g = h^0(5C) \geq 8$ and if $d = 7$ we get $9 = p_g = h^0(7C) \geq 11$, against the assumptions. □

7.1. **Exclusion of $d = 5$, $p_g = 6$, $C^2 = 1$, $K_SC = 5$.** The exclusion of this case is more involved than the previous ones and requires some intermediate steps.

In this case the canonical image Σ is a cone of degree 4 in $\mathbb{P}^5$, so $K_S = 4C + Z$ where $Z > 0$ is contracted to the vertex of Σ. Observe that $5 = K_SC = 4C^2 + CZ = 4 + CZ$ implies $CZ = 1$. We have two cases according to whether the base point p of $|C|$ belongs to Z or not. We start by ruling out the first case.

Lemma 7.2. *In case $p_g = 6$ and $K_SC = 5$, $C^2 = 1$ of Proposition 5.3, the point p does not belong to Z.*

Proof. Assume by contradiction that $p \in Z$. Then for general $C \in |C|$, $K_S|_C = \mathcal{O}_C(5p)$. The linear series $|K_S|_C|$ is base point free, since by assumption the canonical map sends C 5-to-1 onto a line. So p is not a base point of $|K_S|$ and Proposition 6.5 gives $h^0(4C) = 6 = p_g$. Hence Z is contained in the fixed part of $|K_S|$, against the assumption that the general canonical curve is smooth. □

Lemma 7.3. *In case $p_g = 6$ and $K_SC = 5$, $C^2 = 1$ of Proposition 5.3, the curve Z is isomorphic to $\mathbb{P}^1$, $Z^2 = -3$ and Z intersects any curve of $|C|$ transversally at one point distinct from p.*

Proof. We claim that Z is 2-connected. Let $Z = A + B$ be a decomposition with $A, B > 0$. Since C is nef and $CZ = 1$ we may assume $CA = 1$, $CB = 0$. Then we obtain a decomposition $K_S = (4C + A) + B$ and by the 2-connectedness of canonical divisors $2 \leq (4C + A)B = AB$ as claimed.

Since $p \notin Z$ by Lemma 7.2 and $ZC = 1$, there is no curve of $|C|$ that contains Z. So $h^0(\mathcal{O}_Z(C)) \geq 2$. By [CFM97, Prop. A.5.ii] we conclude that Z has no component Γ such that $\Gamma C = 0$. Hence Z is irreducible. Then $CZ = 1$ and $h^0(\mathcal{O}_Z(C)) \geq 2$ means that Z is a smooth rational curve.

The adjunction formula gives $-2 = K_SZ+Z^2 = 4+2Z^2$, namely $Z^2 = -3$. □

Lemma 7.4. *The curves of $|C|$ are 2-connected.*

Proof. Remark that any curve $C \in |C|$ is 1-connected by [Me96, Lem.2.6].

Assume by contradiction that there is a curve $C' \in |C|$ decomposing as $C' = A+B$ with $AB = 1$. Let B be minimal with respect to $B(C'-B) = 1$. Then, by [CFM97, Prop. A.4.ii], B is 2-connected.

Now we claim that A and B have no common components.

Write $A = A_0 + \Delta$, $B = B_0 + \Delta$ where $A_0, B_0, \Delta \geq 0$ and A_0 and B_0 have no common components and assume by contradiction that $\Delta \neq 0$. Then we have $1 = AB = \Delta^2 + \Delta A_0 + \Delta B_0 + A_0B_0 = \Delta(C' - \Delta) + A_0B_0$. Since $A_0B_0 \geq 0$, we have $\Delta(C; -\Delta) \leq 1$. So we conclude by the minimality of B with respect to $B(C' - B) = 1$ that $B_0 = 0$ and $\Delta = B < A$. So $C' = A + B = A_0 + 2B$.

Since $C^2 = 1$ and C is nef necessarily $CA = 1, CB = 0$. But then by 2-connectedness of the canonical divisors necessarily $BZ \geq 1$, and this contradicts $ZC = 1$ (recall that Z is not a component of C by Lemma 7.3).

Hence A and B have no common components and therefore intersect transversely at a single point q.

The point q is thus a disconnecting node of C', and as such a base point of $|K_S + C| = |5C + Z|$. Since q, being a double point of a curve in $|C|$, is distinct from p, we conclude that q lies in Z but this is a contradiction since Z is not a component of any curve in $|C|$ and $ZC = 1$.

So no curve of $|C|$ can decompose as $C = A + B$ with $AB = 1$. □

As in the proof of Lemma 6.7 we let $\bar{S}$ be the surface obtained by blowing up $p \in S$ and contracting the (-2)-curves contained in the curves of $|C|$ (recall that these curves do not pass through p). Let $\bar{f}\colon \bar{S} \to \mathbb{P}^1$ be the fibration induced by $|C|$: the strict transform $\bar{E}$ of the exceptional curve of the blow up and the strict transform $\bar{Z}$ of Z are disjoint sections of $\bar{f}$ and the fibers of $\bar{f}$ are either irreducible or have two components, a component that meets $\bar{E}$ and a component that meets $\bar{Z}$.

Note that, although the fibers of $\bar{f}$ are Gorenstein, it is possible that their components are not so. In order to deal with this situation, we recall the notion of numerically m-connected Gorenstein curve ([CFHR99, Def. 3.1]), that generalizes the analogous definition for curves on a smooth surface. A Gorenstein curve D is numerically m-connected if for any generically Gorenstein subcurve $D' \subset D$ the inequality $\deg \omega_D|_{D'} - (2p_a(D') - 2) \geq m$ holds.

Lemma 7.5. *Let $\bar{C}$ be any fiber of $\bar{f}$. Then $\omega_{\bar{C}}$ is very ample.*

Proof. As explained above, $\bar{C}$ is either irreducible or has two components $\bar{A}$ and $\bar{B}$ such that $\mathcal{O}_{\bar{A}}(K_{\bar{C}}) = \mathcal{O}_{\bar{A}}(K_{\bar{S}}) = \mathcal{O}_{\bar{A}}(5p)$ and $\mathcal{O}_{\bar{B}}(K_{\bar{C}}) = \mathcal{O}_{\bar{B}}(K_{\bar{S}}) = \mathcal{O}_{\bar{B}}(q)$, where p denotes the point $\bar{C} \cap \bar{E}$ and q denotes the point $\bar{C} \cap \bar{Z}$. If we are in the latter case the curve $\bar{C}$ is 2-connected by [CFHR99, Lem. 4.2] and by Lemma 7.4, so $\deg \omega_{\bar{C}}|_{\bar{B}} - 2p_a(\bar{B}) + 2 = 3 - 2p_a(\bar{B}) \geq 2$, hence $p_a(\bar{B}) = 0$.

Since $\bar{A}$ and $\bar{B}$ have no common components one has the following sequence (cf. [CFPR23, Lem. 2.4]):

$$0 \to \omega_{\bar{A}} \to \omega_{\bar{C}} \to \omega_{\bar{C}}|_{\bar{B}} \to 0,$$

that gives $\deg \omega_{\bar{C}} = 2p_a(\bar{A}) - 2 + \deg \omega_{\bar{C}}|_{\bar{B}}$ and also, switching the roles of $\bar{A}$ and $\bar{B}$, $\deg \omega_{\bar{C}} = 2p_a(\bar{B}) - 2 + \deg \omega_{\bar{C}}|_{\bar{A}}$. Equating these two expressions we obtain $7 - 2p_a(\bar{A}) = \deg \omega_{\bar{C}}|_{\bar{A}} - 2p_a(\bar{A}) + 2 = \deg \omega_{\bar{C}}|_{\bar{B}} - 2p_a(\bar{B}) + 2 = 3$, hence $p_a(\bar{A}) = 2$ and $\bar{C}$ is numerically 3-connected. So [CFHR99, Thm. 3.6] applies and either $\omega_{\bar{C}}$ is very ample or $\bar{C}$ is honestly hyperelliptic, i.e., there is a finite degree two map $\psi\colon \bar{C} \to \mathbb{P}^1$ (an "honest g^1_2"). By the above discussion, if $\bar{C}$ is reducible then it cannot be honestly hyperelliptic because $p_a(\bar{A}) \neq p_a(\bar{B})$, while a reducible honestly hyperelliptic curve consists of two components both isomorphic to $\mathbb{P}^1$. If $\bar{C}$ is irreducible then $5p + q$ is a canonical divisor. If $\bar{C}$ were honestly hyperelliptic, then the canonical map would be composed with ψ, hence $p + q$ and $2p$ would be linearly equivalent, contradicting $p \neq q$ and $p_a(\bar{C}) > 0$. □

Lemma 7.6. *Let $\bar{C}$ be any fiber of $\bar{f}$. Then $h^0(\mathcal{O}_{\bar{C}}(3p)) = 1$.*

Proof. By Lemma 7.5, for any pair of smooth points $x, y \in \bar{C}$ one has $h^0(\mathcal{O}_{\bar{C}}(x + y)) = 1$. So, in particular $h^0(\mathcal{O}_{\bar{C}}(2p)) = h^0(\mathcal{O}_{\bar{C}}(p + q)) = 1$ and $h^0(\mathcal{O}_{\bar{C}}(3p)) = h^0(\mathcal{O}_{\bar{C}}(2p + q)) \leq 2$, where the last equality is given by Serre duality, since $5p + q$ is a canonical divisor of $\bar{C}$.

Assume $h^0(\mathcal{O}_{\bar{C}}(3p)) = h^0(\mathcal{O}_{\bar{C}}(2p + q)) = 2$. By the previous arguments $|\mathcal{O}_{\bar{C}}(3p)|$ and $|\mathcal{O}_{\bar{C}}(2p + q)|$ are base point free g^1_3's and are distinct since

$p \neq q$. Let $h\colon \bar{C} \to \mathbb{P}^1 \times \mathbb{P}^1$ be the corresponding morphism and let D be the image.

If $\bar{C}$ is irreducible, then h is birational and D is of type $(3, 3)$. By adjunction $p_a(D) = 4 = p_a(\bar{C})$, hence h is an isomorphism. On the other hand, both g^1_3's are ramified at p, hence $h(p) \in D$ is singular while $p \in \bar{C}$ is smooth. This contradiction shows that $h^0(\mathcal{O}_{\bar{C}}(3p)) = 1$ when $\bar{C}$ is irreducible.

Consider now the case when $\bar{C}$ is reducible. We have seen in the proof of Lemma 7.5 that in this case $\bar{C} = \bar{A} \cup \bar{B}$ with $\bar{A}$, $\bar{B}$ irreducible with $p_a(\bar{A}) = 2$ and $p_a(\bar{B}) = 0$ and such that $p \in \bar{A}$ and $q \in \bar{B}$. The free linear series $|\mathcal{O}_{\bar{C}}(2p + q)|$ and$|\mathcal{O}_{\bar{C}}(3p)|$ restrict to $|\mathcal{O}_{\bar{A}}(2p)|$ and $|\mathcal{O}_{\bar{A}}(3p)|$. Hence both $|\mathcal{O}_{\bar{A}}(2p)|$ and $|\mathcal{O}_{\bar{A}}(3p)|$ are free, but this is impossible. This last contradiction completes the proof. □

We can now exclude this case.

Proposition 7.7. *The case $p_g = 6$ and $K_S C = 5$, $C^2 = 1$ of Proposition 5.3 does not occur.*

Proof of Proposition 7.7. Arguing as in the proof of Lemma 6.7 one shows that for all $C \in |C|$ for every $m \in \mathbb{N}$ we have the equality $h^0(\mathcal{O}_C(mp)) = h^0(\mathcal{O}_{\bar{C}}(mp))$, where $\bar{C}$ is the strict transform of C in $\bar{S}$. So Lemma 7.6 gives $h^0(\mathcal{O}_C(3p)) = 1$ for all C and Corollary 6.4 gives $h^0(\mathcal{O}_S(4C)) = 6 = p_g(S)$. So Z is contained in the fixed part of $|K_S| = |4C + Z|$, contradicting the assumption that the general canonical curve is smooth. □

7.2. Exclusion of $d = 5$, $p_g = 5$, $C^2 = 2$, $K_S = 3C$.

7.2.1. *Preliminary considerations.*

Proposition 7.8. *In the assumptions of Theorem 3.1, if $d = 5$, $p_g = 5$, $C^2 = 2$ then $q(S) = 0$.*

Proof. We write $q := q(S)$ for simplicity.
Step 1: $q \leq 3$.
For C general consider the sequence:

$$0 \to H^0(K_S) \to H^0(K_S + C) \to H^0(K_C) \to H^1(K_S) \to H^1(K_S + C) = 0,$$

where the last term is zero by Kawamata-Viehweg vanishing, since C is nef and big. By assumption the image of the restriction map $H^0(K_S) \to H^0(K_S|_C)$ is 2-dimensional. Since C is a linear pencil, then the image of $H^0(K_S + C) \to H^0(K_C)$ has dimension ≥ 2, so $q = h^1(K_S) \leq g(C) - 2 = 3$.

Step 2: $q \leq 1$.
Assume $q = 2$ or 3. Fix a smooth C. Since C is nef and big the map $\mathrm{Pic}^0(S) \to J(C)$ is actually an injection (see for instance [CFM97, Prop. 1.6]). So the image of the map $\phi_1\colon \mathrm{Pic}^0(S) \to J^2(C)$ defined by $\alpha \mapsto (C + \alpha)|_C$ is isomorphic to an abelian subvariety $A \subset J(C)$ of dimension q. The locus W_2 of effective classes in $J^2(C)$ is a surface birational to the symmetric product

S^2C. So if $q = 3$ then W_2 cannot contain the image of ϕ_1 by dimension reasons, and if $q = 2$ it cannot contain it because W_2 is of general type while the image of ϕ_1 is an abelian surface. So $h^0((C+\alpha)|_C) = 0$ for α general. The exact sequence

$$0 \to H^0(\alpha) \to H^0(C+\alpha) \to H^0((C+\alpha)|_C)$$

gives $h^0(C+\alpha) = 0$ for $\alpha \in \mathrm{Pic}^0(S)$ general. Consider now the map $\phi_2\colon \mathrm{Pic}^0(S) \to J^4(C)$ defined by $\alpha \mapsto (2C+\alpha)|_C$. The image of ϕ_2 is invariant under translation by A, hence if $q = 3$ by [DF93, Proposition 3.3] it cannot be contained in the set of effective classes of $J^4(C)$, which is a translate of the Θ-divisor of $J(C)$. As a consequence, we have $h^0((2C+\alpha)|_C) = 0$ for α general, and as above we get $h^0(2C+\alpha) = 0$. If $q = 2$, [DF93, Proposition 3.3] again implies that $h^0((2C+\alpha)|_C) \leq 1$, and therefore $h^0(2C+\alpha) \leq 1$ for α general.

Using Riemann-Roch on S we compute:

$$4 - q = \chi(C+\alpha) \leq h^0(C+\alpha) + h^0(2C-\alpha), \quad \alpha \text{ general.}$$

We have shown that the right hand side is zero if $q = 3$ and is ≤ 1 if $q = 2$, so we get a contradiction in either case.

Step 3: $q = 0$. By Step 2 it is enough to rule out $q = 1$. So assume by contradiction that $q = 1$ and write $E := \mathrm{Pic}^0(S) = \mathrm{Alb}(S)$ and let $f\colon C \to E$ be the restriction of the Albanese map. If $\phi_1(E)$ is not contained in W_2, then we obtain a contradiction as in Step 2. So assume that ϕ_1 maps E to $W_2 = S^2(C)$. Then by [CMP14, Corollary 3.9] there is a double cover $g\colon C \to E$ such that $\phi_1 = g^*\colon E \to J^1(C)$. Both f and g induce the same injection of E into $J(C)$, so f and g coincide up to a translation in E.

If F is a general fiber of the Albanese map $a\colon S \to E$, then $FC = 2$, and therefore the linear pencil $|C|$ induces a g^1_2 on F. So F is hyperelliptic and the canonical map factors through the hyperelliptic involution of F, contradicting the assumption that the degree of the canonical map is 5, odd. So we conclude that $q = 0$. □

Recall that we assume that the general canonical curve is smooth and the canonical map of X is 5-to-1 onto the cone $\Sigma_3 \subset \mathbb{P}^4$ over the twisted cubic curve of $\mathbb{P}^3$. The curves of $|C|$ are mapped 5-to-1 to the rulings of the cone. By Proposition 5.3, $K_S = 3C$.

Lemma 7.9. *In the assumptions of Theorem 3.1, if $d = 5$, $p_g = 5$, $C^2 = 2$ then*

(i) *The general C is smooth and non hyperelliptic*
(ii) *If $C = A + B$, with $A, B > 0$ then $AB > 0$ is even*
(iii) *$|K_S|$ and $|C|$ have exactly one common base point $p \in S$*
(iv) *The base scheme of $|K_S|$ consists of p, a point p_1 infinitely near to p and a point p_2 infinitely near to p_1*
(v) *$|C|$ has a second base point q, possibly infinitely near to p, in which case $q \neq p_1$*

(vi) *If Z is an irreducible (-2)-curve of X, then Z does not go through p or q and is contained in exactly one curve of $|C|$*

(vii) *Assume that there is an irreducible curve Γ such that $\Gamma C = 1$ and $|C - 2\Gamma|$ is not empty. Then q is infinitely near to p, Γ is unique and 2Γ is contained in the unique curve C_0 in $|C|$ singular at p.*

Proof. (i) By Bertini's theorem any linear system without fixed part and with selfintersection at most 3 has a smooth element. If the general C were hyperelliptic, then the canonical map of C would not separate two points in the same orbit of the hyperelliptic involution. A fortiori, neither $H^0(S, K_S + C)$ nor $H^0(S, K_S)$ would separate those points. This implies that the canonical map of S restricted to C has even degree, a contradiction.

(ii) As $K_S = 3C$, the $2-$connectedness of the canonical divisors implies $AB > 0$. By the genus formula $A(K_S + A) = 4A^2 + 3AB$ is even, and therefore AB is even too.

(iii) As $C^2 = 2$, $|C|$ has two simple base points, possibly infinitely near. The canonical system $|K_S|$ has $(3C)^2 - (\deg \varphi_K)(\deg \varphi_K(S)) = 3$ base points, possibly infinitely near. As $K_S = 3C$ every base point of $|K_S|$ on S is also a base point of $|C|$. In particular $|C|$ and $|K_S|$ have at least a common base point p on S. However, if the whole base scheme of $|C|$, of length 2, were contained in the base scheme of $|K_S|$, then the restriction to C of the canonical system of S would also have base locus of length at least 2, and then the canonical map of S restricted to C would have degree at most $6 - 2 = 4$ a contradiction.

(iv)-(v) Then $|K_S|$ has only one base point on S, that we call p, and the remaining two base points p_1, p_2 are infinitely near to p. The second base point of C, that we call q, if infinitely near to p, cannot be p_1 or p_2 as we have just shown that the base scheme of $|C|$ is not contained in the base scheme of $|K_S|$. Consider the blow up $\beta_p \colon S_p \to S$ with exceptional divisor E_p. The movable part of $|K_{S_p}|$, $|\beta_p^* K_S - E_p|$, has intersection 1 with E_p, so it cannot have two distinct base points on E_p. This shows that p_2 is infinitely near to the p_1.

(vi) This follows immediately by $CZ = 0$.

(vii) As $\Gamma < C$, the condition $\Gamma C = 1$ forces Γ to contain either p or q. If q is not infinitely near to P, then every C in $|C|$ is smooth at p and q, a contradiction. Otherwise, there is exactly one curve C_0 that is singular at p, so $\Gamma \subset C_0$. The uniqueness of Γ is now obvious. □

7.2.2. *Linear systems on the curves of $|C|$.* Let $\epsilon \colon \tilde{S} \to S$ be the blow-up at p and q and let E_p, E_q be the corresponding (-1)-curves, where E_p is reducible if and only if q is infinitely near to p, in which case $A := E_p - E_q$ is an irreducible (-2)-curve.

We denote by $f\colon \tilde{S} \to \mathbb{P}^1$ the fibration induced by $|C|$, with fibre $F := \epsilon^*C - E_p - E_q$. Each fibre F maps to a curve C, and the map is an isomorphism unless q is infinitely near to p and C is the unique element C_0 of $|C|$ singular at p. Then the fibre, say F_0, splits as $F_0 = C_0' + A$, where C_0' is the strict transform of C_0. In addition $p_a(C_0') = 5 - 1 = 4$ and $AC_0' = 2$.

For all curves F in $|F|$ we will denote by p, respectively q, the line bundle restriction of E_p, respectively E_q to F. For F general, we can see p and q as simple points, the intersection of F with E_p, respectively E_q.

In this section we study the linear systems $|ap + bq|$ on the curves C.

By adjunction, we have for every curve of $|F|$:

$$K_{\tilde{S}}|_F = K_F = 4(p+q), \quad \epsilon^*K_S|_F = 3(p+q), \quad |K_S|_F = p + |V|,$$

where $|V| \subset |2p + 3q|$ is a pencil. There is exactly one curve F_0 for which $|V|$ is not base point free, the unique curve F_0 through p_1. If q is infinitely near to p, then F_0 is the already mentioned curve containing A, mapping to the curve C_0 having a double point at p.

Lemma 7.10. *For every $F \in |F|$ we have:*
  (i) *$h^0(F, 3(p+q)) = 3$, $h^0(F, p+q) = 1$;*
  (ii) *$h^0(F, 2(p+q)) = h^0(F, 2p+3q) - 1 \leq 2$.*

Proof. (i) By Serre duality, $h^1(\mathcal{O}_{\tilde{S}}(E_p + E_q)) = h^1(\varepsilon^*K_S)$. In addition $h^1(\varepsilon^*K_S) = h^1(K_S) = 0$, where the first equality holds because $R^1\epsilon_*\mathcal{O}_{\tilde{S}} = 0$ and the second one because of Proposition 7.8. So the sequence

$$0 \to H^0(E_p + E_q) \to H^0(\epsilon^*C) \to H^0(F, p+q) \to 0$$

is exact for every $F \in |F|$ and $h^0(F, p+q) = 1$. Then Riemann-Roch gives $h^0(F, 3(p+q)) = 3$.

(ii) The curve E_q meets every fiber $F \in |F|$ at a smooth point, which is not a base point of $\varepsilon^*|K_S|$ by Lemma 7.9. A fortiori, q is not a base point of $|2p+3q|$, namely $h^0(F, 2(p+q)) = h^0(F, 2p+3q) - 1$. The inequality $h^0(F, 2p+3q) \leq h^0(F, 3(p+q))$ is obvious except for the case when q is infinitely near to p and F is the fiber F_0 dominating the unique curve C_0 in $|C|$ singular at p. In that case, let $s \in H^0(S, A)$ be a non-zero section and consider the map $\mu\colon H^0(F_0, 2p+3q) \to H^0(F_0, 3p+2q)$ defined by multiplying by s. Any section $\sigma \in \ker\mu$ is supported on A and so vanishes on $B|_A$, where $B = F_0 - A$. By Lemma 7.9, $B|_A$ is a scheme of length at least 2, while $2p + 3q$ has degree 1 on A. So $\sigma = 0$, μ is injective and $h^0(F_0, 2p+3q) \leq h^0(F_0, 3p+2q)$. In turn, $h^0(F_0, 3p+2q) \leq h^0(F_0, 3p+3q)$ since E_q meets F_0 at a smooth point. So $h^0(F, 2p+3q) \leq h^0(F, 3(p+q))$ for any $F \in |F|$ and $h^0(F, 2p+3q) \leq 3$ by (i). □

We look now at $|2(p+q)|$, which is a theta characteristic on F.

Lemma 7.11. *There are the following possibilities:*
  (E) $h^0(F, 2(p+q)) = 2$ for every $F \in |F|$ ("even case");
  (O) $h^0(F, 2(p+q)) = 1$ for every $F \in |F|$ ("odd case");

(VO) q is infinitely near to p, $h^0(F, 2(p+q)) = 1$ for every $F \neq F_0$, $h^0(F_0, 2(p+q)) = 2$ and $C_0 \geq 2\Gamma$, where Γ is an irreducible curve such that $C\Gamma = 1$ ("very odd case").

Proof. Recall that $h^0(F, 2(p+q)) \leq 2$ by Lemma 7.10. If $h^0(F, 2(p+q)) = 2$ for F general, then $h^0(F, 2(p+q)) = 2$ for all F by semicontinuity.

So assume that $h^0(F, 2(p+q)) = 1$ for F general. By Lemma 7.9 if a fiber $\bar{F} \in |F|$ contains a multiple irreducible component that is not a (-2)-curve then q is infinitely near to p and $\bar{F} = F_0 = \varepsilon^* C_0 - E_p - E_q$. Then we can apply Harris Theorem 2.5 as in Lemma 6.7 and obtain that the parity of $h^0(F, 2(p+q))$ is constant as F varies in $|F| \setminus \{F_0\}$. So, either we are in case (O) or $h^0(F_0, 2(p+q)) = 2$, and the latter case occurs only if $C_0 \geq 2\Gamma$, where Γ is an irreducible curve with $C\Gamma = 1$. □

7.2.3. *Exclusion of cases (E) and (O) of Lemma 7.11.*

Proposition 7.12. *Case (E) in Lemma 7.11 does not occur.*

Let us first introduce the idea of the proof below. The movable part of the canonical system of $\tilde{S}$ is $|D| = |3F + 2E_p + 3E_q|$. We will see that f_*D is a vector bundle on $\mathbb{P}^1$ of rank 3. To have the canonical image equal to the cone over a twisted cubic curve, we expect $f_*D = \mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{L}$ with $h^0(\mathcal{L}) = 0$. We will prove that such a decomposition holds but $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^1}(1)$, so $p_g = 7 \neq 5$. In other words, the map on the cone would not be given by the whole canonical system, but by a proper subsystem.

Proof. We assume by contradiction that we are in case (E) of Lemma 7.9, so $h^0(F, 2(p+q)) = 2$ for all F. In this case $h^0(F, 2p+3q) = 3$ for all F by Lemma 7.10.

This implies q not infinitely near to p.

In fact, otherwise, choose a general $F \cong C$. Note that $p = q$ on it, we have $h^0(C, 4p) = 2$, $h^0(C, 5p) = 3$, and then, since 5 is a prime number, $|5p|$ defines a morphism $C \to \mathbb{P}^2$ onto a quintic. Since by Riemann-Roch $h^0(C, 3p) = 3 - 5 + 1 + h^0(C, 5p) = 2$, then the quintic has a singular (cuspidal) branch at (the image of) p.

We set local coordinates x, y at p so that p maps to $(0,0)$ and the line $y = 0$ is the one cutting $5p$. So the local equation of the quintic is of the form $x^5 = ya(x,y)$. Since the origin is a singular point that is not an ordinary double point (having a singular branch) then $a(x,y)$ is in the ideal generated by x^2 and y and we can rewrite the quintic as

$$x^5 = y(x^2 b(x,y) + yc(x,y)).$$

Performing a standard blow up $y \mapsto xy$ we obtain $x^5 = xy(x^2 b(x,xy) + xyc(x,xy))$ from which, dividing by x^2

$$x^3 = y(xb(x,xy) + yc(x,xy)).$$

This equation is singular at the origin. So the quintic has at least two singular points, the image of p and a point infinitely near to it. Since the arithmetic genus of a plane quintic is 6 then its normalization has genus ≤ 4, whereas C has genus 5, a contradiction.

Now we show that $\underline{h^0(F, p+2q) = 1 \text{ for all } F}$.

Since q is not infinitely near to p, p and q are distinct points in all curves $C \cong F$. Since $h^0(F, 2p+3q) = 3$, we claim that $|2p+3q|$ separates p and q. Otherwise, since $h^0(F, 2p+2q) = 2$ for all F, there is a curve F such that, on F, p is a base point of $|2p+2q|$. On the other hand, by Riemann-Roch $h^0(F, 2p+q) = 3-5+1+h^0(F, 2p+3q) = 2$ and so q is also a base point of $|2p+2q|$. By $p \neq q$ we deduce $h^0(F, p+q) = 2$ contradicting Lemma 7.10.

Now we set $D = 3F + 2E_p + 3E_q$.

Recall that we have proved that E_p and E_q are two disjoint sections of the fibration $f\colon \tilde{S} \to \mathbb{P}^1$. Since $D \cdot E_p = 1$ and $D \cdot E_q = 0$, the exact sequence

$$0 \to D - E_p - E_q \to D \to D_{|E_p+E_q} \to 0$$

pushes forward to

$$\begin{aligned}(7.1)\quad 0 \to f_*(3F + E_p + 2E_q) \to f_*(3F + 2E_p + 3E_q) \to \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1} \to \\ \to R^1 f_*(3F + E_p + 2E_q) \to R^1 f_*(3F + 2E_p + 3E_q) \to 0\end{aligned}$$

The map among the R^1's is an isomorphism by the same argument used in the proof of Proposition 6.3 about the exact sequence (6.2), since, for all F, $h^0(F, p+2q) = 1$, $h^0(F, 2p+3q) = 3$ and then $h^1(F, p+2q) = h^1(F, 2p+3q) = 2$. As there we deduce a short exact sequence of locally free sheaves

$$(7.2)\quad 0 \to f_*(3F + E_p + 2E_q) \to f_*(3F + 2E_p + 3E_q) \to \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1} \to 0$$

Since $h^0(E_p + 2E_q) = 1$ the line bundle $f_*(E_p + 2E_q)$ is trivial and therefore $f_*(3F + E_p + 2E_q) = f_*(E_p + 2E_q) \otimes \mathcal{O}_{\mathbb{P}^1}(3) = \mathcal{O}_{\mathbb{P}^1}(3)$.

Then (7.2) implies $h^0(3F + 2E_p + 3E_q) = 7$. This contradicts $p_g = 5$ since $|3F + 2E_p + 3E_q|$ is the movable part of the canonical system of $\tilde{S}$. □

Proposition 7.13. *Case (O) of Lemma 7.11 does not occur.*

The idea of the proof is similar to the previous one.

Proof. We argue by contradiction. The assumption is that $h^0(F, 2(p+q)) = 1$ for all F.

We show that $\underline{q \text{ is infinitely near to } p}$.

Otherwise E_p and E_q are disjoint sections of f. Set $D = 3F + 3E_p + 3E_q$ and consider the exact sequence

$$0 \to D - E_p - E_q \to D \to D_{|E_p+E_q} \to 0$$

Since $D \cdot E_p = D \cdot E_q = 0$, this sequence pushes forward to

$$\begin{aligned}(7.3)\quad 0 \to f_*(3F + 2E_p + 2E_q) \to f_*(3F + 3E_p + 3E_q) \to \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \to \\ \to R^1 f_*(3F + 2E_p + 2E_q) \to R^1 f_*(3F + 3E_p + 3E_q) \to 0\end{aligned}$$

Since all $h^q(F, 2p+2q)$ and $h^q(F, 3p+3q)$ do not depend on F, all sheaves in (7.3) are locally free. Arguing as in the proof of Proposition 7.12 we obtain the short exact sequence of vector bundles

$$0 \to f_*(3F + 2E_p + 2E_q) \to f_*(3F + 3E_p + 3E_q) \to \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \to 0 \tag{7.4}$$

and $f_*(3F + 2E_p + 2E_q)$ has rank 1. Since $h^0(\tilde{S}, 2E_p + 2E_q) = 1$ then $f_*(3F+2E_p+2E_q) = f_*(2E_p+2E_q) \otimes \mathcal{O}_{\mathbb{P}^1}(3) = \mathcal{O}_{\mathbb{P}^1}(3)$. Then (7.4) implies $h^0(3F + 3E_p + 3E_q) = 6$ contradicting $p_g = 5$. So q is infinitely near to p.

Consider now the exact sequence

$$0 \to 3F + 2E_p + 2E_q \to 3F + 2E_p + 3E_q \to (3F + 2E_p + 3E_q)_{|E_q} \to 0$$

It pushes forward to the exact sequence of locally free sheaves

$$\begin{aligned} 0 \to f_*(3F + 2E_p + 2E_q) \to f_*(3F + 2E_p + 3E_q) \to \mathcal{O}_{\mathbb{P}^1} \to \\ \to R^1 f_*(3F + 2E_p + 2E_q) \to R^1 f_*(3F + 2E_p + 3E_q) \to 0 \end{aligned} \tag{7.5}$$

By Lemma 7.10 all $h^q(F, 2p + 3q)$ do not depend on F, so, again by the argument of the proof of Proposition 7.12, we get a short exact sequence of locally free sheaves

$$0 \to f_*(3F + 2E_p + 2E_q) \to f_*(3F + 2E_p + 3E_q) \to \mathcal{O}_{\mathbb{P}^1} \to 0 \tag{7.6}$$

and $f_*(3F + 2E_p + 2E_q) \cong \mathcal{O}_{\mathbb{P}^1}(3)$.

Since $Ext^1(\mathcal{O}_{\mathbb{P}^1}, f_*(3F + 2E_p + 2E_q)) = H^1(\mathcal{O}_{\mathbb{P}^1}(3)) = 0$ we deduce from (7.6) the isomorphism

$$f_*(3F + 2E_p + 3E_q) \cong \mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1} \tag{7.7}$$

Observe that by $h^0(F, 2p + 2q) = 1$ it follows $h^0(F, p + 2q) = 1$ for all F: this follows, for $F = F_0$, by the argument at the end of the proof of Lemma 7.10, whereas it is obvious for $F \neq F_0$. Then by Serre Duality and Riemann-Roch Theorem, for all F, $h^1(F, 3p+2q) = 1$ and $h^0(F, 3p+2q) = 2$.

The multiplication by an equation of $A = E_p - E_q$ gives an injective morphism $f_*(3F+2E_p+3E_q) \to f_*(3F+3E_p+2E_q)$ between vector bundles of rank 2, that is, since $3F + 3E_p + 2E_q$ is subcanonical, an isomorphism on global sections. By (7.7) then also

$$f_*(3F + 3E_p + 2E_q) \cong \mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1} \tag{7.8}$$

Finally we push forward the short exact sequence

$$0 \to 3F + 3E_p + 2E_q \to 3F + 3E_p + 3E_q \to (3F + 3E_p + 3E_q)_{|E_q} \to 0$$

obtaining the exact sequence of locally free sheaves

$$\begin{aligned} 0 \to f_*(3F + 3E_p + 2E_q) \to f_*(3F + 3E_p + 3E_q) \to \mathcal{O}_{\mathbb{P}^1} \to \\ \to R^1 f_*(3F + 3E_p + 2E_q) \to R^1 f_*(3F + 3E_p + 3E_q) \to 0 \end{aligned}$$

and then, as in the proof of Proposition 7.12, the short exact sequence

$$0 \to f_*(3F + 3E_p + 2E_q) \to f_*(3F + 3E_p + 3E_q) \to \mathcal{O}_{\mathbb{P}^1} \to 0$$

By (7.8), $h^1(f_*(3F + 3E_p + 2E_q)) = 0$, and then

$$p_g = h^0(3F + 3E_p + 3E_q) = h^0(3F + 3E_p + 2E_q) + h^0(\mathcal{O}_{\mathbb{P}^1}) = p_g + 1.$$

□

7.2.4. *Exclusion of case (VO) of Lemma 7.11.* We now know, by Lemma 7.11 and Propositions 7.12 and 7.13, that if S is a surface in the case $p_g = 5$, $K_S C = 6$ and $C^2 = 2$ of Proposition 5.3 , then the pencil $|C|$ contains a curve $C_0 = 2\Gamma + R$, where Γ is irreducible with $\Gamma C = 1$ and all the components of R are (-2)-curves.

The first base point p of $|C|$ is smooth for Γ. We note that (by $\Gamma C = 1$) the second base point q of $|C|$, infinitely near to p, does not belong to the strict transform of Γ.

We use Γ to construct a new surface Y with an irreducible pencil $|G|$ such that $G^2 = 1$, $K_Y = 7G$ and canonical map of degree 5. We write $C_0 = 2M + \Delta$, where $M, \Delta > 0$ and Δ is reduced. Since $\Gamma \le M$ and $2 = C^2 = 2MC + C\Delta$, we see that $MC = 1$ and Δ is a union of (-2)-curves. Set $L := M + \Delta$, pick $C_1 \in |C|$ general and let $\pi\colon X \to S$ be the double cover given by the relation $2L \sim_{lin} C_1 + \Delta$. So the branching locus of π is the union of C_1 and Δ.

Lemma 7.14. *The divisor Δ is a disjoint union of (-2)-curves $\Delta_1, \dots \Delta_r$.*

Proof. The divisor Δ is reduced and supported on (-2)-curves; we decompose Δ as a sum of disjoint connected divisors $\Delta_1, \dots \Delta_r$. Since S is of general type, for $i = 1, \dots r$ the dual graph of Δ_i is a tree. On the other hand, if A is a component of Δ_i then $0 = AC = 2AM + A\Delta = 2AM + A\Delta_i$, so for all components A of Δ_i the intersection number $A\Delta_i$ is even. This is possible only if Δ_i is irreducible. □

Let $\eta\colon S \to \bar{S}$ be the contraction of $\Delta_1, \dots \Delta_r$ to singular points $s_1, \dots s_r$ of type A_1 and let $|\bar{C}|$ be the pencil of $\bar{S}$ induced by $|C|$. The Stein factorization of $\eta \circ \pi$ gives rise to the following commutative diagram:

(7.9)
$$\begin{array}{ccc} X & \xrightarrow{\ \eta_0\ } & Y \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\bar{\pi}} \\ S & \xrightarrow{\ \eta\ } & \bar{S} \end{array}$$

where X and Y are smooth, π is a flat double cover branched on $C_1 + \Delta$, $\bar{\pi}$ is a double cover branched on the image $\bar{C}_1$ of C_1 and on the points $s_1, \dots s_r \in \bar{S}$, and η_0 is the blow up of the preimages $y_1, \dots y_r$ of $s_1, \dots s_r \in \bar{S}$. We denote by E_i the exceptional curve of X corresponding to y_i (so $\pi^*\Delta_i = 2E_i$), by $x \in X$ the preimage of the base point p of $|C|$ and by y the image point $\eta_0(x)$. Note that, since every (-2)-curve of S does not contain p, then y is a smooth point of Y.

Lemma 7.15. *In the above set-up there is a linear pencil $|D|$ on X with $D^2 = 1$ and base point x such that $\pi^*C = 2D$ and there is a pencil $|G|$ on Y with $G^2 = 1$ and base point y such that $\eta_0^*G = D$, $\pi^*\bar{C} = 2G$.*
In addition, $K_X = 7D + E_1 + \cdots + E_r$ and $K_Y = 7G$.

Proof. Let $C \in |C|$ be a curve distinct from C_0 and C_1. The restriction of π to C is the double cover of C associated to the relation $2L|_C \sim_{lin} 2p$, hence π^*C has an ordinary double point at x. The normalization of π^*C is the double cover given by the relation $2(L|_C(-p)) \sim_{lin} 0$, because $L|_C = \Gamma|_C = \mathcal{O}_C(p)$, and therefore it is the disjoint union of two copies of C. Therefore $\pi^*C = D_1 + D_2$, where D_1 and D_2 are isomorphic to C and meet transversally at x. The monodromy of a loop around the image point $O \in \mathbb{P}^1$ of D_0 exchanges D_1 and D_2, which belong then to the same irreducible pencil, obtained varying C in $|C|$. As its self-intersection is positive (equal to 1) the pencil is linear, and we denote it by $|D|$. The covering involution fixes two curves of $|D|$, namely the preimage D_1 of C_1 and the preimage D_0 of C_0. Since the η_0-exceptional curves are contained in D_0 and do not contain the base point x of $|D|$, $|D|$ induces a pencil $|G|$ of Y with $G^2 = 1$ and $\bar{\pi}^*\bar{C} = 2G$.

The formulae for K_X and K_Y follow from the Hurwitz formula, since $\pi^*K_X = \pi^*(3C) = 6D$ and $\bar{\pi}^*K_Y = \bar{\pi}^*(3\bar{C}) = 6G$. □

In view of diagram (7.9) and of Lemma 7.15, we have $h^i(\mathcal{O}_X(mD)) = h^i(\mathcal{O}_Y(mG))$ for all $i, m \geq 0$. We are mainly interested in Y, but it is often easier to compute these invariants on X, since the double cover π is flat and one can use the projection formulae.

Lemma 7.16. *In the above setup, $8 \leq h^0(6G) \leq 9$ and y is a base point of $|6G|$.*

Proof. Since $6D = \pi^*3C$, $\pi_*\mathcal{O}_X(6D) = \mathcal{O}_S(3C) \oplus \mathcal{O}_S(3C - L)$ by the projection formula, so $h^0(\mathcal{O}_X(6D)) = h^0(\mathcal{O}_S(3C)) + h^0(\mathcal{O}_S(3C - L)) = 5 + h^0(\mathcal{O}_S(3C - L))$. The cohomology group $H^0(\mathcal{O}_S(3C - L))$ is the kernel of the restriction map $H^0(3C) \to H^0(3C|_L)$ and so has dimension ≤ 4 since $|3C|$ has no fixed part. On the other hand $h^0(3C - L) \geq h^0(2C) = 3$, so $8 \leq h^0(\mathcal{O}_X(6D)) \leq 9$.

The decomposition of $\pi_*\mathcal{O}_X(6D)$ induces a decomposition $H^0(\mathcal{O}_X(6D)) = \pi^*H^0(\mathcal{O}_S(3C)) \oplus \sigma\pi^*H^0(\mathcal{O}_S(3C - L))$, where $\sigma \in H^0(\mathcal{O}_X(\pi^*L))$ is a section with zero locus equal to the ramification locus $D_1 + E_1 + \cdots + E_r$ of π. So x is a base point of $|6D|$ and y is a base point of $|6G|$. □

Let $\tilde{Y}$ be the blow up of Y at y, let E_y be the exceptional curve and let $\tilde{G}$ be the general fiber of the fibration $\tilde{g}\colon \tilde{Y} \to \mathbb{P}^1$ induced by $|G|$. For $i = 0, 1$ we denote by G_i, resp. $\tilde{G}_i$, D_i, the pull back of C_i to Y, resp. $\tilde{Y}$, X.

We are going to study in detail the following exact sequence for $m \geq 1$:

$$0 \to \tilde{g}_*mE_y \to \tilde{g}_*(m+1)E_y \to \mathcal{O}_{\mathbb{P}^1}(-m-1) \to \tag{7.10}$$
$$\to R^1\tilde{g}_*mE_y \to R^1\tilde{g}_*(m+1)E_y \to 0$$

Lemma 7.17. *In the above setup:*

(i) *the set of gaps for y in G is $\{1,2,3,4,9\}$ for all $G \in |G| \setminus \{G_0\}$;*
(ii) *the set of gaps for G_0 is $\{1,3,5,7,9\}$.*

Proof. (i) If $G \neq G_0$, then G is isomorphic to a curve $D \in |D| \setminus D_0$ and the isomorphism sends y to p, so $h^0(G, 4y) = 1$ by Lemma 7.11 and the set of gaps is as stated.

(ii) Assume for contradiction that $h^0(G_0, 4y) = 1$. Then $h^0(G, my)$ is independent of $G \in |G|$ and all the sheaves in sequence (7.10) are locally free for all $m \geq 1$. In particular for $0 \leq m \leq 4$, $g_* mE_y \cong \mathcal{O}_{\mathbb{P}^1}$ as it is a line bundle with space of global sections of dimension 1. Moreover for $m \geq 4$ the last map in (7.10) is a surjective morphism of line bundles, hence it is an isomorphism. So the sequence

$$0 \to \tilde{g}_* mE_y \to \tilde{g}_*(m+1)E_y \to \mathcal{O}_{\mathbb{P}^1}(-m-1) \to 0$$

is exact. For $m = 4$ we obtain $\tilde{g}_* 5E_y = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-5)$, since $H^1(\mathcal{O}_{\mathbb{P}^1}(5)) = 0$ and, similarly, $\tilde{g}_* 6E_y = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-5) \oplus \mathcal{O}_{\mathbb{P}^1}(-6)$. Twisting by $\mathcal{O}_{\mathbb{P}^1}(6)$ gives $h^0(6G) = h^0((\tilde{g}_* 6E_y)(6)) = 10$, contradicting Lemma 7.16. So $h^0(G_0, 4y) > 1$.

Applying Mumford-Harris theorem on the parity of theta characteristic as in Lemma 6.7 we see that $h^0(G_0, 4y) = 3$. Arguing as in Lemma 6.6 one shows that the set of gaps is as stated. □

Finally we are ready to exclude this case, by studying the torsion part of the sheaves $R^1\tilde{g}_* mE_y$. So denote by τ_m the torsion subsheaf of $R^1\tilde{g}_* mE_y$. By Lemma 7.17 and by cohomology and base change, τ_m is supported, for all m, on the image point $O \in \mathbb{P}^1$ of $\tilde{G}_0$. More precisely, $\tau_m \neq 0$ if and only if $2 \leq m \leq 6$ and τ_m is a cyclic module except for $m = 4$, in which case, by Theorem 2.6, $\tau_4 \cong \tau \oplus \tau$ with τ cyclic.

Proposition 7.18. *The case $p_g = 5$, $K_S C = 6$ and $C^2 = 2$ of Proposition 5.3 does not occur.*

Proof. As in the proof of Lemma 7.17, (ii), for $0 \leq m \leq 4$, $g_* mE_y \cong \mathcal{O}_{\mathbb{P}^1}$.

We consider again sequence (7.10) for $m = 4$. The map $R^1\tilde{g}_* 4E_y \to R^1\tilde{g}_* 5E_y$ is an isomorphism modulo torsion, since it is a surjective map of rank 1 sheaves, but it is not an isomorphism since τ_4 and τ_5 are not isomorphic, as remarked above. So (7.10) splits in two short exact sequences

$$0 \to \tilde{g}_* 4E_y \to \tilde{g}_* 5E_y \to \mathcal{O}_{\mathbb{P}^1}(-5-a) \to 0 \tag{7.11}$$

$$0 \to \mathbb{C}[t]/t^a \to R^1\tilde{g}_* 4E_y \to R^1\tilde{g}_* 5E_y \to 0 \tag{7.12}$$

where $a > 0$ is an integer. The exact sequence (7.11) yields $\tilde{g}_* 5E_y = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-5-a)$ and twisting by $\mathcal{O}_{\mathbb{P}^1}(5)$ we obtain $h^0(5G) = 6$ and also, by Lemma 6.2, $h^0(mG) = m+1$ for $1 \leq m \leq 4$. Twisting by $\mathcal{O}_{\mathbb{P}^1}(6)$ we obtain

$h^0(6\tilde{G}+5E_y) \leq 7+h^0(\mathcal{O}_{\mathbb{P}^1}(1-a)) \leq 8$. On the other hand, $h^0(6\tilde{G}+5E_y) = h^0(6G) \geq 8$ by Lemma 7.16, so $h^0(6G) = 8$:

$$a = 1 \qquad\qquad \tilde{g}_* 5E_y \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-6)$$

Since both $R^1\tilde{g}_*4E_y$ and $R^1\tilde{g}_*5E_y$ have rank 1, (7.12) forces an exact sequence among the torsion subsheaves

$$0 \to \mathbb{C}[t]/t \to \tau_4 \to \tau_5 \to 0 \tag{7.13}$$

As τ, τ_5 are cyclic modules, and $\tau_4 = \tau \oplus \tau$,

$$\tau \cong \tau_5 \cong \mathbb{C}[t]/t$$

By relative duality $R^1\tilde{g}_*5E_y \cong \mathcal{O}_{\mathbb{P}^1}(-9) \oplus \tau_5$, and then $h^0(R^1\tilde{g}_*(5\tilde{G}+5E_y)) = h^0(\tau_5) = 1$. By the Leray spectral sequence $h^1(5\tilde{G}+5E_y) = h^1(\tilde{g}_*(5\tilde{G}+5E_y)) + h^0(R^1\tilde{g}_*(5\tilde{G}+5E_y)) = 1$. On the other hand $h^0(5\tilde{G}+5E_y) = h^0(5G) = 6$ and $h^2(5\tilde{G}+5E_y) = h^0(2\tilde{G}+3E_y) = h^0(2\tilde{G}+2E_y) = h^0(2G) = 3$. So $\chi(5\tilde{G}+5E_y) = 6-1+3 = 8$ which implies, by the standard Riemann-Roch Theorem for surfaces

$$\chi(\mathcal{O}_Y) = 8 + \frac{1}{2}(5\tilde{G}+5E_y)(2\tilde{G}+3E_y) = 13$$

and then $h^0(7G) = p_g(Y) \geq 12$.

Now consider the exact sequence

$$0 \to H^0(6G) \to H^0(7G) \to H^0(\mathcal{O}_G(7y)),$$

with $G \in |G|$ a general curve. The vector spaces have respective dimensions 8, ≥ 12 and 4, and then the restriction map $H^0(7G) \to H^0(\mathcal{O}_G(7y))$ is surjective. Therefore y is not a base point of $|K_Y|$, and then Proposition 6.5 forces $h^0(5G) \geq 7$, contradicting $h^0(5G) = 6$. □

References

[BHPV04] W. Barth, K. Hulek, C. Peters, A. Van De Ven, *Compact complex surfaces, 2nd enlarged ed.* Ergebnisse der Mathematik und ihrer Grenzgebiete, **3.** Folge, Band **4**, Springer-Verlag, Berlin (2004).

[Be79] A. Beauville, *L'application canonique pour les surfaces de type général.* Inv. Math. **55** (1979), 121–140.

[Bo73] E. Bombieri, *Canonical models of surfaces of general type*, Inst. Hautes Études Sci. Publ. Math., **42** (1973), 171–219.

[CFHR99] F. Catanese, M. Franciosi, K. Hulek, M. Reid, *Embeddings of curves and surfaces*, Nagoya Math. J. **154** (1999), 185–220.

[CDM25] C. Ciliberto, T. Dedieu, M. Mendes Lopes, *On surfaces of high degree with respect to the sectional genus*, preprint arXiv:2503.22299.

[CFM97] C. Ciliberto, P. Francia, M. Mendes Lopes, *Remarks on the bicanonical map for surfaces of general type*, Math. Z. **224** (1997), 137–166.

[CFPR23] S. Coughlan, M. Franciosi, R. Pardini, S. Rollenske, *Half canonical rings of Gorenstein spin curves of genus two*, preprint arXiv:2312.09671.

[CMP14] C. Ciliberto, M. Mendes Lopes, R. Pardini, *Abelian varieties in Brill–Noether loci*, Adv. Math. **257** (2014), 349–364.

[Co89] M. Cornalba, *Moduli of curves and theta-characteristics*, Lectures on Riemann surfaces (Trieste, 1987), 560–589. World Scientific Publishing Co., Inc., Teaneck, NJ, 1989

[DF93] O. Debarre, R. Fahlaoui, *Abelian Varieties in $W^r_d(C)$ and points of bounded degree on algebraic curves*, Compositio Math. **88** (1993), no. 3, 235–249.

[Ha82] J. Harris, *Theta-Characteristics on Algebraic Curves*, Transactions of the American Mathematical Society, Jun., 1982, Vol. **271**, No. 2 (Jun., 1982), pp. 611–638

[KM08] K. Konno, M. Mendes Lopes *The base components of the dualizing sheaf of a curve on a surface*, Archiv der Mathematik, **90** (2008), 395–400.

[Me96] M. Mendes Lopes, *Adjoint systems on surfaces*, Boll. Un. Mat. Ital. A (7) **10** (1996), no. 1, 169–179.

[MP98] M. Mendes Lopes, R. Pardini, *Triple canonical surfaces of minimal degree*, International Journal of Mathematics, **152** (1998), 203–230.

[MP23] M. Mendes Lopes, R. Pardini, *On the degree of the canonical map of a surface of general type*, The Art of Doing Algebraic Geometry, Dedieu, Thomas (ed.) et al., Birkhäuser. Trends Math., 305-325 (2023).

[MPP26] M. Mendes Lopes, R. Pardini, R. Pignatelli, *The parity of theta characteristics is preserved by infinitesimal deformations*, arXiv: 2604.07876, to appear.

[Mu71] D. Mumford, *Theta characteristics of an algebraic curve.* Ann. Sci. École Norm. Sup. (4) **4** (1971), 181–192.

[Re86] M. Reid, *Surfaces of Small Degree*, Math. Ann. **275**, 71–80 (1986).

[St98] F. Starnone, *Superfici algebriche con curve canoniche*, Boll. Unione Mat. Ital., Sez. A, Mat. Soc. Cult. (8) 1, Suppl., 67-69 (1998).

[Xi86] G. Xiao, *Algebraic surfaces with high canonical degree*, Math. Ann. **274**, 473–483 (1986).

[Xi87] G. Xiao, *Irregularity of surfaces with a linear pencil*, Duke Math. J. **55** 3 (1987), 596–602.

Margarida Mendes Lopes
Centro de Análise Matemática,
Geometria e Sistemas Dinâmicos,
Departamento de Matemática
Instituto Superior Técnico
Universidade de Lisboa
Av. Rovisco Pais, 1
1049-001 Lisboa, Portugal
mmendeslopes@tecnico.ulisboa.pt

Rita Pardini
Dipartimento di Matematica
Università di Pisa
Largo B. Pontecorvo, 5
56127 Pisa, Italy
rita.pardini@unipi.it

Roberto Pignatelli
Dipartimento di Matematica
Università di Trento
via Sommarive 14
38123 Trento, Italy
roberto.pignatelli@unitn.it